VERTEX OPERATOR ALGEBRAS, EXTENDED E_8 DIAGRAM, AND MCKAY'S OBSERVATION ON THE MONSTER SIMPLE GROUP

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Abstract. We study McKay's observation on the Monster simple group, which relates the $2A$-involutions of the Monster simple group to the extended E_8 diagram, using the theory of vertex operator algebras (VOAs). We first consider the sublattices L of the E_8 lattice obtained by removing one node from the extended E_8 diagram at each time. We then construct a certain coset (or commutant) subalgebra U associated with L in the lattice VOA $V_{\sqrt{2}E_8}$. There are two natural conformal vectors of central charge $1/2$ in U such that their inner product is exactly the value predicted by Conway [1]. The Griess algebra of U coincides with the algebra described in [1, Table 3]. There is a canonical automorphism of U of order $|E_8/L|$. Such an automorphism can be extended to the Leech lattice VOA V_Λ and it is in fact a product of two Miyamoto involutions. In the sequel [12] to this article we shall develop the representation theory of U. It is expected that if U is actually contained in the Moonshine VOA $V^\natural$, the product of two Miyamoto involutions is in the desired conjugacy class of the Monster simple group.

1. Introduction

The Moonshine vertex operator algebra $V^\natural$ constructed by Frenkel-Lepowsky-Meurman [7] is one of the most important examples of vertex operator algebras (VOAs). Its full automorphism group is the Monster simple group. The weight 2 subspace $V_2^\natural$ of $V^\natural$ has a structure of commutative non-associative algebra which coincides with the 196884-dimensional algebra investigated by Griess [9] in his construction of the Monster simple group (see also Conway[1]). The structure of this algebra, which is called the Monstrous Griess algebra, has been studied by group theorists. It is well known [1] that each $2A$-involution ϕ of the Monster simple group uniquely defines an idempotent e_ϕ called an axis in the Monstrous Griess algebra. Moreover, the inner product $\langle e_\phi, e_\psi\rangle$ of any two axes e_ϕ and e_ψ is uniquely determined by the conjugacy class of the product $\phi\psi$ of $2A$-involutions. Actually, $2A$-involutions of the Monster simple group satisfy a 6-transposition property, that is, $|\,\phi\psi\,| \le 6$ for any two $2A$-involutions ϕ and ψ. In addition, the conjugacy class of $\phi\psi$ is one of $1A$, $2A$, $3A$, $4A$, $5A$, $6A$, $4B$, $2B$, or $3C$.

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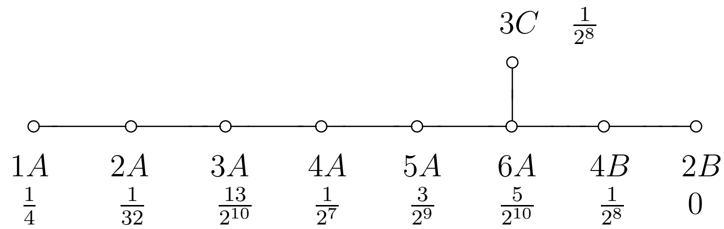
John McKay [14] observed that there is an interesting correspondence with the extended E_8 diagram. Namely, one can assign $1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B$, and $3C$ to the nodes of the extended E_8 diagram as follows (cf. Conway [1], Glauberman and Norton [8]):



(1.1)

where the numerical labels are equal to the multiplicities of the corresponding simple roots in the highest root and the numbers behind the labels denote the inner product $\langle 2e_\phi, 2e_\psi\rangle$ of $2e_\phi$ and $2e_\psi$.

On the other hand, from the point of view of VOAs, Miyamoto [15, 17] showed that an axis is essentially a half of a conformal vector e of central charge $1/2$ which generates a Virasoro VOA $\mathrm{Vir}(e) \cong L(1/2, 0)$ inside the Moonshine VOA $V^\natural$. Moreover, an involutive automorphism τ_e can be defined by

$$\tau_e = \begin{cases} 1 & \text{on } W_0 \oplus W_{1/2}, \\ -1 & \text{on } W_{1/16}, \end{cases}$$

where W_h denotes the sum of all irreducible $\mathrm{Vir}(e)$-modules isomorphic to $L(1/2, h)$ inside $V^\natural$. In fact, τ_e is always of class $2A$ for any conformal vector e of central charge $1/2$ in $V^\natural$.

In this article, we try to give an interpretation of the McKay diagram (1.1) using the theory of VOAs. We first observe that there is a conformal vector $\hat{e}$ of central charge $1/2$ in the lattice VOA $V_{\sqrt{2}E_8}$ which is fixed by the action of the Weyl group of type E_8. Let Φ be the root system corresponding to the Dynkin diagram obtained by removing one node from the extended E_8 diagram and $L = L(\Phi)$ the root lattice associated with Φ. Then the Weyl group $W(\Phi)$ of Φ and the quotient group E_8/L both act naturally on $V_{\sqrt{2}E_8}$ and their actions commute with each other. The action of the quotient group E_8/L can be extended to the Leech lattice VOA V_Λ.

The main idea is to construct certain vertex operator subalgebras U of the lattice VOA $V_{\sqrt{2}E_8}$ corresponding to the nine nodes of the McKay diagram. In each case, U is constructed as a coset (or commutant) subalgebra of $V_{\sqrt{2}E_8}$ associated with Φ. In fact, U is chosen so that the Weyl group $W(\Phi)$ acts trivially on it. We show that in each of the nine cases U always contains $\hat{e}$ and another conformal vector $\hat{f}$ of central charge $1/2$ such that the inner product $\langle \hat{e}, \hat{f}\rangle$ is exactly the value listed in the McKay diagram. Both of $\hat{e}$ and $\hat{f}$ are fixed by the Weyl group $W(\Phi)$. Thus the Miyamoto involutions $\tau_{\hat{e}}$ and $\tau_{\hat{f}}$ commute with the action of $W(\Phi)$. Furthermore, the quotient group E_8/L naturally induces some automorphism of U of order $n = |E_8/L|$, which is identical with the numerical label of the corresponding node in the McKay diagram. Such an automorphism can be extended to the Leech lattice VOA V_Λ and it is in fact a product $\tau_{\hat{e}}\tau_{\hat{f}}$ of two Miyamoto involutions $\tau_{\hat{e}}$ and $\tau_{\hat{f}}$.

In the sequel [12] to this article we shall study the properties of the coset subalgebra U in detail. Except the $4A$ case, U always contains a set of mutually orthogonal conformal vectors such that their sum is the Virasoro element of U and the central charge of those conformal vectors are all coming from the unitary series

$$c = c_m = 1 - \frac{6}{(m+2)(m+3)}, \qquad m = 1, 2, 3, \ldots.$$

Such a conformal vector generates a Virasoro VOA isomorphic to $L(c_m, 0)$ inside U. The structure of U as a module for a tensor product of those Virasoro VOA is determined.

In the $4A$ case, U is isomorphic to the fixed point subalgebra $V_{\mathcal{N}}^{+}$ of θ for some rank two lattice $\mathcal{N}$, where θ is an automorphism of $V_{\mathcal{N}}$ induced from the -1 isometry of the lattice $\mathcal{N}$.

The VOA U is generated by $\hat{e}$ and $\hat{f}$. As a consequence we know that every element of U is fixed by the Weyl group $W(\Phi)$. The weight 1 subspace U_1 of U is 0. The Griess algebra U_2 of U is also generated by $\hat{e}$ and $\hat{f}$ and it has the same structure as the algebra studied in Conway [1, Table 3]. The automorphism group of U is a dihedral group of order $2n$ except the cases for $1A$, $2A$, and $2B$. It is a trivial group in the $1A$ case, a symmetric group of degree 3 in the $2A$ case, and of order 2 in the $2B$ case. Furthermore, we shall discuss the rationality of U and the classification of irreducible modules. The product $\tau_{\hat{e}}\tau_{\hat{f}}$ of two Miyamoto involutions should be in the desired conjugacy class of the Monster simple group, provided that the Moonshine VOA $V^{\natural}$ contains a subalgebra isomorphic to U.

Further mysteries concerning the McKay diagram can be found in Glauberman and Norton [8]. Among other things, some relation between the Weyl group $W(\Phi)$ and the centralizer of a certain subgroup generated by two $2A$-involutions and one $2B$-involution in the Monster simple group was discussed. That every element of U is fixed by $W(\Phi)$ seems quite suggestive.

Let us recall some terminology (cf. [7]). A VOA is a $\mathbb{Z}$-graded vector space $V = \oplus_{n\in\mathbb{Z}} V_n$ with a linear map $Y(\,\cdot\,, z) : V \to (\operatorname{End} V)[[z, z^{-1}]]$ and two distinguished vectors; the vacuum vector $\mathbf{1} \in V_0$ and the Virasoro element $\omega \in V_2$ which satisfy certain conditions. For any $v \in V$, $Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ is called a vertex operator and $v_n \in \operatorname{End} V$ a component operator. Each homogeneous subspace V_n is the eigenspace for the operator $L(0) = \omega_1$ with eigenvalue n. The eigenvalue for $L(0)$ is called a weight. Suppose $V = \oplus_{n=0}^{\infty} V_n$ with $V_0 = \mathbb{C}\mathbf{1}$ and $V_1 = 0$. For $u, v \in V_2$, one can define a product $u \cdot v$ by $u_1 v$ and an inner product $\langle u, v\rangle$ by $u_3 v = \langle u, v\rangle \mathbf{1}$. The inner product is invariant, that is, $\langle u_1 v, w\rangle = \langle v, u_1 w\rangle$ for $u, v, w \in V_2$ (cf. [7, Section 8.9]). With the product and the inner product V_2 becomes an algebra, which is called the Griess algebra of V.

The organization of the article is as follows. In Section 2 we review some notation for lattice VOAs from [7] and certain conformal vectors in the lattice VOA $V_{\sqrt{2}R}$ given by [5], where R is a root lattice of type A, D, or E. Moreover, we study some highest weight vectors in irreducible modules of $V_{\sqrt{2}R}$ with respect to those conformal vectors. In Section 3 we consider the sublattice L of E_8 and define the coset subalgebra U and two conformal vectors $\hat{e}$ and $\hat{f}$ of central charge 1/2. We calculate the inner product $\langle \hat{e}, \hat{f}\rangle$ and verify that it is identical with the value given in the McKay diagram. A canonical automorphism σ of order $n = |E_8/L|$ induced by the quotient group E_8/L is also discussed. Then in

Section 4 we consider an embedding of an orthogonal sum $\sqrt{2}E_8{}^3$ of three copies of $\sqrt{2}E_8$ into the Leech lattice Λ and show that the product $\tau_{\hat{e}}\tau_{\hat{f}}$ of two Miyamoto involutions $\tau_{\hat{e}}$ and $\tau_{\hat{f}}$ is of order n as an automorphism of V_Λ. Finally, in Section 5 we give an explicit correspondence between the Griess algebra U_2 of U and the algebra in Conway [1, Table 3].

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2. Conformal vectors in lattice VOAs

In this section, we review the construction of certain conformal vectors in the lattice VOA $V_{\sqrt{2}R}$ from [5], where R is a root lattice of type A_n, D_n, or E_n. The notation for lattice VOAs here is standard (cf. [7]). Let N be a positive definite even lattice with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$. Then the VOA V_N associated with N is defined to be $M(1)\otimes\mathbb{C}\{N\}$. More precisely, let $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{Z}}N$ be an abelian Lie algebra and $\hat{\mathfrak{h}}=\mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ its affine Lie algebra. Then $M(1)=\mathbb{C}[\alpha(n)\,|\,\alpha\in\mathfrak{h},n<0]\cdot 1$ is the unique irreducible $\hat{\mathfrak{h}}$-module such that $\alpha(n)\cdot 1=0$ for $\alpha\in\mathfrak{h}$, $n\ge 0$ and $K=1$, where $\alpha(n)=\alpha\otimes t^n$. Moreover, $\mathbb{C}\{N\}$ denotes a twisted group algebra of the additive group N. In the case for $N=\sqrt{2}R$, the twisted group algebra $\mathbb{C}\{\sqrt{2}R\}$ is isomorphic to the ordinary group algebra $\mathbb{C}[\sqrt{2}R]$ since $\sqrt{2}R$ is a doubly even lattice. The standard basis of $\mathbb{C}[\sqrt{2}R]$ is denoted by $e^{\sqrt{2}\alpha}$, $\alpha\in R$. Then the vacuum vector $\mathbf{1}$ is $1\otimes e^0$.

Let Φ be the root system of R and Φ^+ and Φ^- the set of all positive roots and negative roots, respectively. Then $\Phi=\Phi^+\cup\Phi^-=\Phi^+\cup(-\Phi^+)$. The Virasoro element ω of $V_{\sqrt{2}R}$ is given by

$$\omega=\omega(\Phi)=\frac{1}{2h}\sum_{\alpha\in\Phi^+}\alpha(-1)^2\cdot 1,$$

where h is the Coxeter number of Φ. Now define

$$\begin{aligned} s=s(\Phi)&=\frac{1}{2(h+2)}\sum_{\alpha\in\Phi^+}\left(\alpha(-1)^2\cdot 1-2(e^{\sqrt{2}\alpha}+e^{-\sqrt{2}\alpha})\right),\\ \tilde{\omega}=\tilde{\omega}(\Phi)&=\omega-s. \end{aligned}\tag{2.1}$$

It is shown in [5] that $\tilde{\omega}$ and s are mutually orthogonal conformal vectors, that is, $\tilde{\omega}_1\tilde{\omega}=2\tilde{\omega}$, $s_1s=2s$, and $\tilde{\omega}_1s=0$. The central charge of $\tilde{\omega}$ is $2n/(n+3)$ if R is of type A_n, 1 if R is of type D_n and $6/7, 7/10$ and $1/2$ if R is of type E_6, E_7 and E_8, respectively.

Let $W(\Phi)$ be the Weyl group of Φ. Any element $g\in W(\Phi)$ induces an automorphism of the lattice R and hence it defines an automorphism of the VOA $V_{\sqrt{2}R}$ by

$$g(u\otimes e^{\sqrt{2}\alpha})=gu\otimes e^{\sqrt{2}g\alpha}\qquad\text{for}\quad u\otimes e^{\sqrt{2}\alpha}\in M(1)\otimes e^{\sqrt{2}\alpha}\subset V_{\sqrt{2}R}.$$

Note that both s and $\tilde{\omega}$ are fixed by the Weyl group $W(\Phi)$.

We shall study certain highest weight vectors with respect to the subalgebra $\mathrm{Vir}(s)\otimes\mathrm{Vir}(\tilde{\omega})$, where $\mathrm{Vir}(s)$ and $\mathrm{Vir}(\tilde{\omega})$ denote the Virasoro VOAs generated by the conformal vectors s and $\tilde{\omega}$, respectively.

Let $R^*=\{\alpha\in\mathbb{Q}\otimes_{\mathbb{Z}}R\,|\,\langle\alpha,R\rangle\subset\mathbb{Z}\}$ be the dual lattice of R.

Lemma 2.1. *Let R be a root lattice of type A, D, or E and $\gamma + R$ a coset of R in R^*. Let $k = \min\{\langle\alpha,\alpha\rangle | \alpha \in \gamma + R\}$. For any $\eta \in \gamma + R$ with $\langle\eta,\eta\rangle = k$, we define*

$$X_\eta = \{(\alpha,\beta) \in R \times (\gamma + R) \mid \langle\alpha,\alpha\rangle = 2, \langle\beta,\beta\rangle = k \text{ and } \alpha + \beta = \eta\}.$$

Then $|X_\eta| = kh$, where h is the Coxeter number of R.

Proof. The proof is just by direct verification. We only discuss the case for $R = A_n$. The other cases can be proved similarly.

Let $R = A_n$. Then the Coxeter number h is $n+1$ and the roots of A_n are given by the vectors in the form $\pm(1,-1,0^{n-1}) \in \mathbb{R}^{n+1}$, that is, the vectors whose one entry is ± 1, another entry is ∓ 1, and the remaining $n-1$ entries are 0. Let $\mu = \frac{1}{n+1}(1,\dots,1,-n)$. Then $\mu + R$ is a generator of the group R^*/R. Denote $\gamma = j\mu$ for $j = 0,\dots,n$. Then

$$k = \min\{\langle\alpha,\alpha\rangle | \alpha \in \gamma + R\} = \frac{j(n+1-j)}{n+1},$$

and the elements of square norm k in $\gamma + R$ are of the form

$$\frac{1}{n+1}(j^{n+1-j}, (-n-1+j)^j).$$

Now it is easy to see that $|X_\eta| = (n+1-j)j = kh$ for any η with $\langle\eta,\eta\rangle = k$. □

Proposition 2.2. *Let $\gamma + R$ be a coset of R in R^* and $k = \min\{\langle\alpha,\alpha\rangle | \alpha \in \gamma + R\}$. Define*

$$v = \sum_{\substack{\alpha \in \gamma + R \\ \langle\alpha,\alpha\rangle = k}} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}(\gamma+R)}.$$

Then v is a highest weight vector of highest weight $(0,k)$ in $V_{\sqrt{2}(\gamma+R)}$ with respect to $\mathrm{Vir}(s) \otimes \mathrm{Vir}(\tilde{\omega})$, *that is, $s_j v = \tilde{\omega}_j v = 0$ for all $j \geq 2$, $s_1 v = 0$, and $\tilde{\omega}_1 v = kv$.*

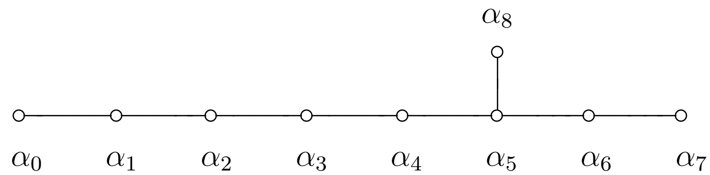
Proof. Since k is the minimum weight of $V_{\sqrt{2}(\gamma+R)}$, it is clear that $s_j v = \tilde{\omega}_j v = 0$ for all $j \geq 2$. Since $\omega_1 v = kv$, it suffices to show that $s_1 v = 0$. By the definition (2.1) of s and the above lemma, we have

$$\begin{aligned} s_1 v &= \frac{1}{2(h+2)} \sum_{\alpha \in \Phi^+} \Big(\alpha(-1)^2 \cdot 1 - 2(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha})\Big)_1 v \\ &= \Big(\frac{h}{h+2}\omega - \frac{1}{h+2} \sum_{\alpha \in \Phi^+} (e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha})\Big)_1 v \\ &= \frac{hk}{h+2} v - \frac{hk}{h+2} v = 0. \end{aligned}$$

Hence the assertion holds. □

3. Extended E_8 diagram and sublattices of the root lattice E_8

In this section, we consider certain sublattices of the root lattice E_8 by using the extended E_8 diagram



(3.1)

where $\alpha_1, \alpha_2, \ldots, \alpha_8$ are the simple roots of E_8 and

$$\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = 0. \tag{3.2}$$

Thus $\langle \alpha_i, \alpha_i \rangle = 2$, $0 \le i \le 8$. Moreover, for $i \ne j$, $\langle \alpha_i, \alpha_j \rangle = -1$ if the nodes α_i and α_j are connected by an edge and $\langle \alpha_i, \alpha_j \rangle = 0$ otherwise. Note that $-\alpha_0$ is the highest root.

For any $i = 0, 1, \ldots, 8$, let $L(i)$ be the sublattice generated by $\alpha_j, 0 \le j \le 8, j \ne i$. Then $L(i)$ is a rank 8 sublattice of E_8. In fact, $L(i)$ is the lattice associated with the Dynkin diagram obtained by removing the corresponding node α_i from the extended E_8 diagram (3.1). Note that the index $|E_8/L(i)|$ is equal to n_i, where n_i is the coefficient of α_i in the left hand side of (3.2). Actually, we have

$$\begin{array}{lll} L(0) \cong E_8, & L(1) \cong A_1 \oplus E_7, & L(2) \cong A_2 \oplus E_6, \\ L(3) \cong A_3 \oplus D_5, & L(4) \cong A_4 \oplus A_4, & L(5) \cong A_5 \oplus A_2 \oplus A_1, \\ L(6) \cong A_7 \oplus A_1, & L(7) \cong D_8, & L(8) \cong A_8. \end{array} \tag{3.3}$$

Remark 3.1. If n_i is not a prime, there is an intermediate sublattice as follows.

$$\begin{array}{ll} A_3 \oplus D_5 \subset D_8 \subset E_8, & \\ A_5 \oplus A_2 \oplus A_1 \subset A_2 \oplus E_6 \subset E_8, & A_5 \oplus A_2 \oplus A_1 \subset A_1 \oplus E_7 \subset E_8, \\ A_7 \oplus A_1 \subset A_1 \oplus E_7 \subset E_8. & \end{array}$$

There are corresponding power maps between conjugacy classes of the Monster simple group, namely,

$$(4A)^2 = 2B, \qquad (6A)^2 = 3A, \qquad (6A)^3 = 2A, \qquad (4B)^2 = 2A,$$

where $(mX)^k = nY$ means that the k-th power g^k of an element g in the conjugacy class mX is in the conjugacy class nY (cf. [2]).

3.1. Coset subalgebras of the lattice VOA $V_{\sqrt{2}E_8}$.

We shall construct some VOAs U corresponding to the nine nodes of the McKay diagram (1.1). In each case, we show that the VOA U contains some conformal vectors of central charge $1/2$ and the inner products among these conformal vectors are the same as the numbers given in the McKay diagram.

Let us explain the details of our construction. First, we fix $i \in \{0, 1, \ldots, 8\}$ and denote $L(i)$ by L. In each case, $|E_8/L| = n_i$ and $\alpha_i + L$ is a generator of the quotient group E_8/L. Hence we have

$$E_8 = L \cup (\alpha_i + L) \cup (2\alpha_i + L) \cup \cdots \cup ((n_i - 1)\alpha_i + L). \tag{3.4}$$

Then the lattice VOA $V_{\sqrt{2}E_8}$ can be decomposed as

$$V_{\sqrt{2}E_8} = V_{\sqrt{2}L} \oplus V_{\sqrt{2}\alpha_i+\sqrt{2}L} \oplus \cdots \oplus V_{\sqrt{2}(n_i-1)\alpha_i+\sqrt{2}L},$$

where $V_{\sqrt{2}j\alpha_i+\sqrt{2}L}$, $j = 0, 1, \ldots, n_i - 1$, are irreducible modules of $V_{\sqrt{2}L}$ (cf. [4]).

The quotient group E_8/L induces an automorphism σ of $V_{\sqrt{2}E_8}$ such that

$$\sigma(u) = \xi^j u \qquad \text{for any} \quad u \in V_{\sqrt{2}j\alpha_i+\sqrt{2}L}, \tag{3.5}$$

where $\xi = e^{2\pi\sqrt{-1}/n_i}$ is a primitive n_i-th root of unity. More precisely, let

$$\mathbf{a} = \begin{cases} \alpha_1 & \text{if } i = 0, \\ -\frac{1}{i+1}(\alpha_0 + 2\alpha_1 + \cdots + i\alpha_{i-1}) & \text{if } 1 \le i \le 5, \\ -\frac{1}{8}(\alpha_0 + 2\alpha_1 + \cdots + 6\alpha_5 + 7\alpha_8) & \text{if } i = 6, \\ \frac{1}{2}(\alpha_6 + \alpha_8) & \text{if } i = 7, \\ -\frac{1}{9}(\alpha_0 + 2\alpha_1 + \cdots + 8\alpha_7) & \text{if } i = 8. \end{cases} \tag{3.6}$$

Then $\langle \mathbf{a}, \alpha_j \rangle \in \mathbb{Z}$ for $0 \le j \le 8$ with $j \ne i$ and $\langle \mathbf{a}, \alpha_i \rangle \equiv -1/n_i \pmod{\mathbb{Z}}$. The automorphism $\sigma : V_{\sqrt{2}E_8} \to V_{\sqrt{2}E_8}$ is in fact defined by

$$\sigma = e^{-\pi\sqrt{-1}\boldsymbol{\beta}(0)} \qquad \text{with} \quad \boldsymbol{\beta} = \sqrt{2}\mathbf{a}. \tag{3.7}$$

For $u \in M(1) \otimes e^\alpha \subset V_{\sqrt{2}E_8}$, we have $\sigma(u) = e^{-\pi\sqrt{-1}\langle \boldsymbol{\beta}, \alpha \rangle} u$. Note that $\mathbf{a} + R$ is a generator of the quotient group R^*/R for the cases $i \ne 0, 7$, where R is an indecomposable component of the lattice L of type A and R^* is the dual lattice of R.

For any lattice VOA V_N associated with a positive definite even lattice N, there is a natural involution θ induced by the isometry $\alpha \to -\alpha$ for $\alpha \in N$. If $N = \sqrt{2}E_8$, which is doubly even, we may define $\theta : V_{\sqrt{2}E_8} \to V_{\sqrt{2}E_8}$ by

$$\alpha(-n) \to -\alpha(-n) \qquad \text{and} \qquad e^\alpha \to e^{-\alpha} \tag{3.8}$$

for $\alpha \in \sqrt{2}E_8$ (cf. [7]). Then $\theta\sigma\theta = \sigma^{-1}$ and the group generated by θ and σ is a dihedral group of order $2n_i$.

Let $R_1, \ldots, R_l$ be the indecomposable components of the lattice L and $\Phi_1, \ldots, \Phi_l$ the corresponding root systems of $R_1, \ldots, R_l$ (cf. (3.3)). Then $L = R_1 \oplus \cdots \oplus R_l$ and

$$V_{\sqrt{2}L} \cong V_{\sqrt{2}R_1} \otimes \cdots \otimes V_{\sqrt{2}R_l},$$

(see [6] for tensor products of VOAs). By (2.1), one obtains $2l$ mutually orthogonal conformal vectors

$$s^k = s(\Phi_k), \quad \tilde{\omega}^k = \tilde{\omega}(\Phi_k), \quad k = 1, \ldots, l \tag{3.9}$$

such that the Virasoro element ω of $V_{\sqrt{2}L}$, which is also the Virasoro element of $V_{\sqrt{2}E_8}$, can be written as a sum of these conformal vectors

$$\omega = s^1 + \cdots + s^l + \tilde{\omega}^1 + \cdots + \tilde{\omega}^l.$$

Now we define U to be a coset (or commutant) subalgebra

$$U = \{v \in V_{\sqrt{2}E_8} \,|\, (s^k)_1 v = 0 \text{ for all } k = 1, \ldots, l\}. \tag{3.10}$$

Note that U is a VOA with the Virasoro element $\omega' = \tilde{\omega}^1 + \cdots + \tilde{\omega}^l$ and the automorphism σ defined by (3.5) induces an automorphism of order n_i on U. By abuse of notation, we denote it by σ also.

Remark 3.2. In [11], it is shown that $\{v \in V_{\sqrt{2}A_n} \mid s(A_n)_1 v = 0\}$ is isomorphic to a parafermion algebra $W_{n+1}(2n/(n+3))$ of central charge $2n/(n+3)$. Thus, if L has some indecomposable component of type A_n, then U contains some subalgebra isomorphic to a parafermion algebra. It is well known [18] that the parafermion algebra $W_{n+1}(2n/(n+3))$ possesses a certain $\mathbb{Z}_{n+1}$ symmetry in the fusion rules among its irreducible modules. The automorphism σ is in fact related to such a symmetry. More details about the relation between coset subalgebra U and the parafermion algebra $W_{n+1}(2n/(n+3))$ can be found in [12].

3.2. **Conformal vectors of central charge** $1/2$**.** Next, we shall study some conformal vectors in $V_{\sqrt{2}E_8}$. We shall also show that the coset subalgebra U always contains some conformal vectors of central charge $1/2$. Moreover, the inner products among these conformal vectors will be discussed.

Recall that the lattice $\sqrt{2}E_8$ can be constructed by using the $[8,4,4]$ Hamming code H_8 and the Construction A (cf. [3]). That means

$$\sqrt{2}E_8 = \left\{(a_1,\dots,a_8) \in \mathbb{Z}^8 \mid (a_1,\dots,a_8) \in H_8 \mod 2\right\}. \tag{3.11}$$

We denote the vectors $(0,0,0,0,0,0,0,0)$ and $(1,1,1,1,1,1,1,1)$ by $\mathbf{0}$ and $\mathbf{1}$, respectively. For any $\gamma \in H_8$, we define

$$X^0_\gamma = \sum_{\substack{\alpha\equiv\gamma \bmod 2\\ \langle\alpha,\alpha\rangle=4}} (-1)^{\langle\alpha,\mathbf{0}\rangle/2} e^\alpha = \sum_{\substack{\alpha\equiv\gamma \bmod 2\\ \langle\alpha,\alpha\rangle=4}} e^\alpha,$$

$$X^1_\gamma = \sum_{\substack{\alpha\equiv\gamma \bmod 2\\ \langle\alpha,\alpha\rangle=4}} (-1)^{\langle\alpha,\mathbf{1}\rangle/2} e^\alpha,$$

and for any binary word $\delta \in {\mathbb{Z}_2}^8$, we define

$$\hat{e}^\epsilon_\delta = \frac{1}{16}\omega + \frac{1}{32}\sum_{\gamma\in H_8} (-1)^{\langle\delta,\gamma\rangle} X^\epsilon_\gamma, \qquad \epsilon = 0,1,$$

where ω is the Virasoro element of the VOA $V_{\sqrt{2}E_8}$. Note that $X^\epsilon_{\mathbf{1}} = 0$ for any $\epsilon = 0,1$ and that $\hat{e}^\epsilon_\delta = \hat{e}^\epsilon_\eta$ if and only if $\eta \in \delta + H_8$

Lemma 3.3. *For any $\epsilon = 0,1$ and $\delta \in {\mathbb{Z}_2}^8$, $\hat{e}^\epsilon_\delta$ is a conformal vector of central charge $1/2$. The inner product among them are as follows.*

$$\langle \hat{e}^\epsilon_\delta, \hat{e}^\epsilon_\eta\rangle = \begin{cases} 0 & \text{if } \delta+\eta \text{ is even} \\ 1/32 & \text{if } \delta+\eta \text{ is odd}\end{cases}$$

for any $\eta \notin \delta + H_8$, and

$$\langle \hat{e}^0_\delta, \hat{e}^1_\eta\rangle = 0$$

for any $\delta,\eta \in {\mathbb{Z}_2}^8$.

Proof. We have

$$(X^\epsilon_\gamma)_1(X^\epsilon_\zeta) = 4X^\epsilon_{\gamma+\zeta} \qquad \text{if} \qquad |\gamma+\zeta| = 4,$$
$$(X^\epsilon_{\mathbf{0}})_1(X^\epsilon_{\mathbf{0}}) = \sum_{\substack{\alpha\equiv\mathbf{0} \bmod 2\\ \langle\alpha,\alpha\rangle=4}} \frac{1}{2}\alpha(-1)^2\cdot 1.$$

Moreover, for any $\gamma \in H_8$ with $|\gamma| = 4$,

$$(X^\epsilon_\gamma)_1(X^\epsilon_\gamma) + (X^\epsilon_{\mathbf{1}+\gamma})_1(X^\epsilon_{\mathbf{1}+\gamma}) = \sum_{\substack{\alpha\equiv\gamma \bmod 2\\ \langle\alpha,\alpha\rangle=4}} \frac{1}{2}\alpha(-1)^2\cdot 1 + \sum_{\substack{\alpha\equiv\mathbf{1}+\gamma \bmod 2\\ \langle\alpha,\alpha\rangle=4}} \frac{1}{2}\alpha(-1)^2\cdot 1 + 8X^\epsilon_{\mathbf{0}}.$$

Note also that

$$\sum_{\gamma\in H_8}\sum_{\substack{\alpha\equiv\gamma \bmod 2\\ \langle\alpha,\alpha\rangle=4}} \frac{1}{2}\alpha(-1)^2\cdot 1 = \sum_{\beta\in\Phi(E_8)}\beta(-1)^2\cdot 1 = 2\sum_{\beta\in\Phi^+(E_8)}\beta(-1)^2\cdot 1.$$

In addition, we have

$$\langle X^\epsilon_\gamma, X^\epsilon_\zeta\rangle = \begin{cases} 16 & \text{if } \gamma=\zeta \text{ and } \langle\gamma,\gamma\rangle \neq 8,\\ 0 & \text{otherwise,}\end{cases}$$

$$\langle X^0_\gamma, X^1_\zeta\rangle = \begin{cases} -16 & \text{if } \gamma=\zeta=0,\\ 0 & \text{otherwise.}\end{cases}$$

Then since $\omega_1\omega = 2\omega$ and $\langle\omega,\omega\rangle = 4$, it follows that

$$\begin{aligned}(\hat{e}^\epsilon_\delta)_1\hat{e}^\epsilon_\delta &= \Big(\frac{1}{16}\omega + \frac{1}{32}\sum_{\gamma\in H_8}(-1)^{\langle\delta,\gamma\rangle}X^\epsilon_\gamma\Big)_1\Big(\frac{1}{16}\omega + \frac{1}{32}\sum_{\gamma\in H_8}(-1)^{\langle\delta,\gamma\rangle}X^\epsilon_\gamma\Big)\\
&= \frac{1}{2^8}\times 2\omega + 2\times\frac{1}{16}\times\frac{1}{32}\times 2\sum_{\gamma\in H_8}(-1)^{\langle\delta,\gamma\rangle}X^\epsilon_\gamma\\
&\quad + \frac{1}{2^{10}}\Big(\sum_{\beta\in\Phi^+(E_8)}2\beta(-1)^2\cdot 1 + 56\sum_{\gamma\in H_8}(-1)^{\langle\delta,\gamma\rangle}X^\epsilon_\gamma\Big)\\
&= \frac{1}{8}\omega + \frac{1}{16}\sum_{\gamma\in H_8}(-1)^{\langle\delta,\gamma\rangle}X^\epsilon_\gamma = 2\hat{e}^\epsilon_\delta,\end{aligned}$$

and

$$\langle\hat{e}^\epsilon_\delta,\hat{e}^\epsilon_\delta\rangle = \frac{1}{2^8}\times 4 + \frac{1}{2^{10}}\times 240 = \frac{1}{4}.$$

Hence $\hat{e}^\epsilon_\delta$ is a conformal vectors of central charge 1/2.

For any $\eta\notin\delta+H_8$, we calculate that

$$\begin{aligned}\langle\hat{e}^\epsilon_\delta,\hat{e}^\epsilon_\eta\rangle &= \frac{1}{2^8}\times 4 + \frac{1}{2^{10}}\sum_{\gamma\in H_8}(-1)^{\langle\delta+\eta,\gamma\rangle}\langle X^\epsilon_\gamma, X^\epsilon_\gamma\rangle\\
&= \begin{cases}\frac{1}{64}+\frac{1}{2^{10}}\times 16\times(7-8) = 0 & \text{if } \delta+\eta \text{ is even,}\\ \frac{1}{64}+\frac{1}{2^{10}}\times 16\times(8-7) = \frac{1}{32} & \text{if } \delta+\eta \text{ is odd.}\end{cases}\end{aligned}$$

Note that there are exactly eight elements in H_8 which are orthogonal to $\delta+\eta$. Note also that $\delta+\eta$ is orthogonal to $(1,1,1,1,1,1,1,1)$ if and only if $\delta+\eta$ is even.

Finally, for any $\delta,\eta\in{\mathbb{Z}_2}^8$ we obtain

$$\langle \hat{e}^0_\delta, \hat{e}^1_\eta\rangle = \frac{1}{2^8}\times 4 - \frac{1}{2^{10}}\times 16 = 0.$$

□

In Miyamoto [16], certain conformal vectors of central charge 1/2 are constructed inside the Hamming code VOA. Our construction of $\hat{e}^\epsilon_\delta$ is essentially a lattice analogue of Miyamoto's construction. In fact, take $\lambda_j=(0,\dots,2,\dots,0)\in\mathbb{Z}^8$ to be the element in $\sqrt{2}E_8$ such that the j-th entry is 2 and all other entries are zero. Then we have a set of 16 mutually orthogonal conformal vectors of central charge 1/2 given by

$$\omega^{\pm}_{\lambda_j} = \frac{1}{16}\lambda_j(-1)^2\cdot 1 \pm \frac{1}{4}(e^{\lambda_j}+e^{-\lambda_j}),\quad j=1,2,\dots,8.$$

A set of mutually orthogonal conformal vectors of central charge 1/2 whose sum is equal to the Virasoro element in a VOA is called a Virasoro frame. Thus, $\{\omega^{\pm}_{\lambda_j}\,|\,1\le j\le 8\}$ is a Virasoro frame of $V_{\sqrt{2}E_8}$. With respect to this Virasoro frame, the lattice VOA $V_{\sqrt{2}E_8}$ is a code VOA (cf. [16]). Let $V^+_{\sqrt{2}E_8}$ be the fixed point subalgebra of $V_{\sqrt{2}E_8}$ under the automorphism θ (cf. (3.8)). Then $\omega^{\pm}_{\lambda_j}\in V^+_{\sqrt{2}E_8}$ and $V^+_{\sqrt{2}E_8}$ is isomorphic to a code VOA M_D, where D is the second order Reed-Müller code $RM(4,2)$ of length 16. Note that $\dim RM(4,2)=11$ and the dual code of $RM(4,2)$ is the first order Reed-Müller code $RM(4,1)$ with the generating matrix

$$\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0\\
1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0
\end{pmatrix}$$

Let H^+ and H^- be the subcodes of D whose supports are contained in the positions corresponding to $\{\omega^+_{\lambda_j}\,|\,1\le j\le 8\}$ and $\{\omega^-_{\lambda_j}\,|\,1\le j\le 8\}$, respectively. Then H^+ and H^- are both isomorphic to the $[8,4,4]$ Hamming code H_8. The conformal vectors $\hat{e}^0_\delta$ and $\hat{e}^1_\delta$ are actually the conformal vectors s_δ constructed by Miyamoto [16] using the Hamming code VOAs M_{H^+} and M_{H^-}, respectively.

Proposition 3.4. *The set* $\{\hat{e}^0_\delta,\hat{e}^1_\zeta\,|\,\delta,\zeta\in{\mathbb{Z}_2}^8/H_8,\delta,\zeta\ \text{are even}\}$ *is a Virasoro frame of* $V^+_{\sqrt{2}E_8}$*. Moreover,* $V^+_{\sqrt{2}E_8}\cong M_{RM(4,2)}$ *with respect to this frame, where* $M_{RM(4,2)}$ *denotes the code VOA associated with the second order Reed-Müller code* $RM(4,2)$*.*

Proof. The first assertion follows from Lemma 3.3. As mentioned above, we know that $V^+_{\sqrt{2}E_8}\cong M_D$ with respect to the frame $\{\omega^{\pm}_{\lambda_j}\,|\,1\le j\le 8\}$, where $D\cong RM(4,2)$. It contains a subalgebra isomorphic to $M_{H^+}\otimes M_{H^-}$. For convenience, we arrange the positions of $\{\omega^{\pm}_{\lambda_j}\}$ so that the support $\operatorname{supp} H^+$ of H^+ is $(1^8,0^8)$ and the support $\operatorname{supp} H^-$ of H^- is $(0^8,1^8)$. Let $\{\beta_0,\beta_1,\dots,\beta_7\}$ with $\beta_0=0$ be a complete set of coset representatives of

$D/(H^+ \oplus H^-)$. Then

$$V^+_{\sqrt{2}E_8} \cong M_{H^+\oplus H^-} \oplus \bigoplus_{i=1}^{7} M_{\beta_i+(H^+\oplus H^-)}.$$

By a result of Miyamoto [16], $M_{H^+\oplus H^-}$ is still isomorphic to the code VOA $M_{H^+\oplus H^-}$ associated with $H^+ \oplus H^-$ with respect to the frame $\{\hat{e}^0_\delta, \hat{e}^1_\zeta \,|\, \delta, \zeta \in {\mathbb{Z}_2}^8/H_8, \delta, \zeta \text{are even}\}$. Moreover, we know that $(1^8, 0^8)$ and $(0^8, 1^8)$ are contained in the dual code of D. Thus $\langle (1^8, 0^8), \beta_i\rangle = \langle (0^8, 1^8), \beta_i\rangle = 0$ for all i. Let β^+ and β^- be such that $\mathrm{supp}\beta^+ \subset \mathrm{supp}H^+$, $\mathrm{supp}\beta^- \subset \mathrm{supp}H^-$, and $\beta_i = \beta^+ + \beta^-$. Then $M_{\beta_i+(H^+\oplus H^-)} \cong M_{\beta^+ + H^+} \otimes M_{\beta^- + H^-}$ and both of $M_{\beta^+ + H^+}$ and $M_{\beta^- + H^-}$ are of integral weight. Hence, by [16], $M_{\beta_i+(H^+\oplus H^-)}$ is again isomorphic to $M_{\beta_i+(H^+\oplus H^-)}$ with respect to the frame $\{\hat{e}^0_\delta, \hat{e}^1_\zeta \,|\, \delta, \zeta \in {\mathbb{Z}_2}^8/H_8, \delta, \zeta \text{are even}\}$ and thus we still have $V^+_{\sqrt{2}E_8} \cong M_D$. □

Now let

$$\begin{aligned} \hat{e} = \hat{e}^0_{\mathbf{0}} &= \frac{1}{16}\omega + \frac{1}{32}\sum_{\alpha\in\Phi^+(E_8)} (e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}), \\ \hat{f} &= \sigma\hat{e}, \end{aligned} \tag{3.12}$$

where σ is the automorphism defined by (3.5). These conformal vectors of central charge 1/2 play an important role for the rest of the paper.

Let Φ be the root system of $L = L(i)$. Let $H_j = \{\alpha \in j\alpha_i + L \,|\, \langle\alpha, \alpha\rangle = 2\}$ be the set of all roots in the coset $j\alpha_i + L$ for $j = 1, \ldots, n_i - 1$. Then

$$\Phi(E_8) = \Phi \cup \bigcup_{j=1}^{n_i-1} H_j.$$

We introduce weight 2 elements X^j, namely,

$$X^j = \sum_{\alpha\in H_j} e^{\sqrt{2}\alpha}, \quad j = 1, \ldots, n_i - 1. \tag{3.13}$$

Then

$$\begin{aligned} \hat{e} &= \frac{1}{16}\omega + \frac{1}{32}\Big(\sum_{\alpha\in\Phi} e^{\sqrt{2}\alpha} + \sum_{j=1}^{n_i-1} X^j\Big), \\ \hat{f} &= \frac{1}{16}\omega + \frac{1}{32}\Big(\sum_{\alpha\in\Phi} e^{\sqrt{2}\alpha} + \sum_{j=1}^{n_i-1} \xi^j X^j\Big), \end{aligned} \tag{3.14}$$

where $\xi = e^{2\pi\sqrt{-1}/n_i}$ is a primitive n_i-th root of unity.

Lemma 3.5. (1) $X^j \in U$, $j = 1, \ldots, n_i - 1$.
(2) $\hat{e}, \hat{f} \in U$.

Proof. Let s^k be defined as in (3.9). Then by a similar argument as in the proof of Proposition 2.2, we can verify that $(s^k)_1 X^j = 0$ and $(s^k)_1\hat{e} = 0$ for $k = 1, \ldots, l$. Thus $X^j, \hat{e} \in U$ by the definition (3.10) of U. Since σ leaves U invariant, we also have $\hat{f} \in U$. □

Remark 3.6. The Weyl group $W(E_8)$ of the root system of type E_8 acts naturally on the lattice VOA $V_{\sqrt{2}E_8}$ and $\hat{e}$ is the only conformal vector among $\hat{e}^0_\delta, \hat{e}^1_\zeta$ which is fixed by $W(E_8)$. The conformal vector $\hat{f}$ is fixed by the Weyl group $W(\Phi) = W(\Phi_1) \times \cdots \times W(\Phi_l)$ of the root system $\Phi = \Phi_1 \oplus \cdots \oplus \Phi_l$ of $L = L(i)$. The conformal vector $\hat{e}$ is also fixed by the automorphism θ (cf. (3.8)). However, $\hat{f}$ is not fixed by θ in general.

Theorem 3.7. *Let $\hat{e}, \hat{f}$ be defined as in* (3.12). *Then*

$$\langle \hat{e}, \hat{f} \rangle = \begin{cases} 1/4 & \text{if } i = 0, \\ 1/32 & \text{if } i = 1, \\ 13/2^{10} & \text{if } i = 2, \\ 1/2^7 & \text{if } i = 3, \\ 3/2^9 & \text{if } i = 4, \\ 5/2^{10} & \text{if } i = 5, \\ 1/2^8 & \text{if } i = 6, \\ 0 & \text{if } i = 7, \\ 1/2^8 & \text{if } i = 8. \end{cases} \tag{3.15}$$

In other words, the values of $\langle \hat{e}, \hat{f} \rangle$ are exactly the values given in McKay's diagram (1.1).

Proof. By (3.14), we can easily obtain that

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}} \Big(|\Phi| + \sum_{j=1}^{n_i - 1} \xi^j |H_j| \Big),$$

where $H_j = \{\alpha \in j\alpha_i + L \mid \langle \alpha, \alpha \rangle = 2\}$.

If $i = 0$, then $n_0 = 1$ and $|\Phi| = 240$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{240}{2^{10}} = \frac{1}{4}.$$

If $i = 1$, then $n_1 = 2$, $|\Phi| = |\Phi(A_1)| + |\Phi(E_7)| = 128$, and $|H_1| = 112$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(128 - 112) = \frac{1}{32}.$$

If $i = 2$, then $n_2 = 3$, $|\Phi| = |\Phi(A_2)| + |\Phi(E_6)| = 78$, and $|H_1| = |H_2| = 81$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(78 - 81) = \frac{13}{2^{10}}.$$

If $i = 3$, then $n_3 = 4$, $|\Phi| = |\Phi(A_3)| + |\Phi(D_5)| = 52$, $|H_1| = |H_3| = 64$, and $|H_2| = 60$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(52 - 60) = \frac{1}{2^7}.$$

If $i = 4$, then $n_4 = 5$, $|\Phi| = |\Phi(A_4)| + |\Phi(A_4)| = 40$, and $|H_1| = |H_2| = |H_3| = |H_4| = 50$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(40 - 50) = \frac{3}{2^9}.$$

If $i = 5$, then $n_5 = 6$, $|\Phi| = |\Phi(A_1)| + |\Phi(A_2)| + |\Phi(A_5)| = 38$, $|H_1| = |H_5| = 36$, $|H_2| = |H_4| = 45$, and $|H_3| = 40$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(38 + 36 - 45 - 40) = \frac{5}{2^{10}}.$$

If $i = 6$, then $n_6 = 4$, $|\Phi| = |\Phi(A_1)| + |\Phi(A_7)| = 58$, $|H_1| = |H_3| = 56$, and $|H_2| = 70$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(58 - 70) = \frac{1}{2^8}.$$

If $i = 7$, then $n_7 = 2$, $|\Phi| = |\Phi(D_8)| = 112$, and $|H_1| = 128$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(112 - 128) = 0.$$

If $i = 8$, then $n_8 = 3$, $|\Phi| = |\Phi(A_8)| = 72$, and $|H_1| = |H_2| = 84$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(72 - 84) = \frac{1}{2^8}.$$

Thus we have proved the theorem. □

Remark 3.8. The same result still holds if we replace $\hat{e}$ by $\hat{e}^{\epsilon}_{\delta}$ and $\hat{f} = \sigma\hat{e}$ by $\sigma\hat{e}^{\epsilon}_{\delta}$ for any $\epsilon = 0, 1$ and $\delta \in {\mathbb{Z}_2}^8$.

4. Miyamoto's τ-involutions and the canonical automorphism σ

Let V be a VOA. If V contains a conformal vector w of central charge $1/2$ such that the subalgebra $\mathrm{Vir}(w)$ generated by w is isomorphic to the Virasoro VOA $L(1/2, 0)$, then an automorphism τ_w of V with $(\tau_w)^2 = 1$ can be defined. Indeed, V is a direct sum of irreducible $\mathrm{Vir}(w)$-modules. Denote by W_h the sum of all irreducible direct summands which are isomorphic to $L(1/2, h)$. Then τ_w is defined to be 1 on $W_0 \oplus W_{1/2}$ and -1 on $W_{1/16}$ (cf. [15, 17]). Thus τ_w is the identity if V has no irreducible direct summand isomorphic to $L(1/2, 1/16)$. We call τ_w the Miyamoto involution or the τ-involution associated with w.

In this section, we shall study the relationship between the canonical automorphism σ and the Miyamoto involutions $\tau_{\hat{e}}$, $\tau_{\sigma\hat{e}}, \dots$, and $\tau_{\sigma^{n_i-1}\hat{e}}$. Let us recall two conformal vectors $\hat{e}$ and $\hat{f}$ of central charge $1/2$ defined by (3.12) and two automorphisms σ and θ introduced in Subsection 3.1.

Lemma 4.1. *As automorphisms of $V_{\sqrt{2}E_8}$, $\tau_{\hat{e}} = \theta$.*

Proof. By Proposition 3.4, we know that $\{\hat{e}^0_{\delta}, \hat{e}^1_{\zeta} \,|\, \delta, \zeta \in {\mathbb{Z}_2}^8/H_8,\ \delta, \zeta \text{ even}\}$ is a Virasoro frame of $V^+_{\sqrt{2}E_8}$ and with respect to this frame, $V^+_{\sqrt{2}E_8}$ is a code VOA isomorphic to $M_{RM(4,2)}$. Therefore, $\tau_{\hat{e}}|_{V^+_{\sqrt{2}E_8}} = \mathrm{id}$. On the other hand,

$$\begin{aligned}\hat{e}_1\gamma(-1)\cdot 1 &= \frac{1}{16}\omega_1\gamma(-1)\cdot 1 + \frac{1}{32}\sum_{\alpha\in\Phi^+(E_8)}(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha})_1\gamma(-1)\cdot 1\\ &= \frac{1}{16}\gamma(-1)\cdot 1\end{aligned}$$

for any $\gamma \in \sqrt{2}E_8$. By the definition of $\tau_{\hat{e}}$, this implies that $\tau_{\hat{e}}(\gamma(-1)\cdot 1) = -\gamma(-1)\cdot 1$. Then $\tau_{\hat{e}}|_{V^-_{\sqrt{2}E_8}} = -\mathrm{id}$, since $V^-_{\sqrt{2}E_8}$ is an irreducible $V^+_{\sqrt{2}E_8}$-module. Hence $\tau_{\hat{e}} = \theta$ as automorphisms of $V_{\sqrt{2}E_8}$. □

Theorem 4.2. *As automorphisms of $V_{\sqrt{2}E_8}$, $\tau_{\hat{e}}\tau_{\hat{f}} = (\sigma^{-1})^2 = e^{2\pi\sqrt{-1}\beta(0)}$ and thus $|\tau_{\hat{e}}\tau_{\hat{f}}| = n_i$ if n_i is odd and $|\tau_{\hat{e}}\tau_{\hat{f}}| = n_i/2$ if n_i is even.*

Proof. Since $\hat{f} = \sigma\hat{e}$, we have $\tau_{\hat{f}} = \sigma\tau_{\hat{e}}\sigma^{-1}$. By (3.5) and the preceding lemma, we also have $\tau_{\hat{e}}\sigma\tau_{\hat{e}} = \theta\sigma\theta = \sigma^{-1}$. Hence the assertion holds by (3.7). □

Next, we shall extend $\tau_{\hat{e}}, \tau_{\hat{f}}$, and σ to the Leech lattice VOA V_Λ. According to the presentation (3.11) of $\sqrt{2}E_8$, the dual lattice $\mathcal{L}$ of $\sqrt{2}E_8$ is given by

$$\mathcal{L} = \{(a_1,\dots,a_8) \in \frac{1}{2}\mathbb{Z}^8 \mid 2(a_1,\dots,a_8) \in H_8 \mod 2\}.$$

Note that $|\mathcal{L}/\sqrt{2}E_8| = 2^8$. Note also that

$$V_{\mathcal{L}} = S(\mathfrak{h}^-_{\mathbb{Z}}) \otimes \mathbb{C}\{\mathcal{L}\} \cong \bigoplus_{\alpha+\sqrt{2}E_8 \in \mathcal{L}/\sqrt{2}E_8} V_{\alpha+\sqrt{2}E_8}$$

as a module of $V_{\sqrt{2}E_8}$.

For any coset $\alpha + \sqrt{2}E_8$ of $\sqrt{2}E_8$ in $\mathcal{L}$, one can always find a coset representative α whose square norm is minimum in the coset such that α is in one of the following forms.

$$\begin{aligned}
&(0^8),\quad (1,0^7),\quad (1^2,0^6),\quad ((1/2)^4,0^4),\\
&((1/2)^3,-1/2,0^4),\quad ((1/2)^2,(-1/2)^2,0^4),\quad ((1/2)^4,1,0^3),\\
&((1/2)^3,-1/2,1,0^3),\quad ((1/2)^8),\quad ((1/2)^7,-1/2),\quad ((1/2)^6,(-1/2)^2).
\end{aligned} \tag{4.1}$$

The square norm $\langle\alpha,\alpha\rangle$ of such α is 0, 1, or 2. Moreover, if $\langle\alpha,\alpha\rangle = 2$, then α can be written as a sum $\alpha = a+b$, where $a, b \in \mathcal{L}$ are in the above forms with $\langle a,a\rangle = \langle b,b\rangle = 1$ and $\langle a,b\rangle = 0$. In particular, the minimal weight of the irreducible module $V_{\alpha+\sqrt{2}E_8}$ is either $1/2$ or 1 for $\alpha \notin \sqrt{2}E_8$.

Now $\sigma = e^{-\pi\sqrt{-1}\beta(0)}$ (cf. (3.7)) acts on $V_{\mathcal{L}}$ as an automorphism of order $2n_i$. The τ-involution $\tau_{\hat{e}}$ also acts on $V_{\mathcal{L}}$. In fact, $V_{\alpha+\sqrt{2}E_8}$ is $\tau_{\hat{e}}$-invariant for any coset $\alpha+\sqrt{2}E_8$ of $\sqrt{2}E_8$ in $\mathcal{L}$.

Lemma 4.3. *For any $x \in \mathcal{L}$ with $\langle x,x\rangle = 1$, $\tau_{\hat{e}}(e^x) = -e^{-x}$.*

Proof. If $\langle\gamma,\gamma\rangle = 4$ and $\langle\gamma+x,\gamma+x\rangle = 1$ for some $\gamma \in \sqrt{2}E_8$, then $\langle\gamma,x\rangle = -2$ and $\gamma + x = -x$. Thus, by the definition of $\hat{e}$ it follows that

$$\hat{e}_1 e^x = \frac{1}{16}\Big(\frac{1}{2}e^x\Big) + \frac{1}{32}e^{-x} \quad\text{and}\quad \hat{e}_1 e^{-x} = \frac{1}{16}\Big(\frac{1}{2}e^{-x}\Big) + \frac{1}{32}e^{x}.$$

Therefore, $\hat{e}_1(e^x + e^{-x}) = \frac{1}{16}(e^x+e^{-x})$ and $\hat{e}_1(e^x - e^{-x}) = 0$. Hence $\tau_{\hat{e}}(e^x+e^{-x}) = -(e^x+e^{-x})$ and $\tau_{\hat{e}}(e^x - e^{-x}) = e^x - e^{-x}$ by the definition of $\tau_{\hat{e}}$, and so $\tau_{\hat{e}}(e^x) = -e^{-x}$. □

Lemma 4.4. *Let $\alpha + \sqrt{2}E_8$ be a coset of $\sqrt{2}E_8$ in $\mathcal{L}$. Then for any $u \in V_{\alpha+\sqrt{2}E_8}$, $\tau_{\hat{e}}\sigma\tau_{\hat{e}}(u) = \sigma^{-1}(u)$.*

Proof. We have $V_{\alpha+\sqrt{2}E_8} = \mathrm{span}_{\mathbb{C}}\{v_n e^\alpha \,|\, v \in V_{\sqrt{2}E_8},\, n \in \mathbb{Z}\}$, since $V_{\alpha+\sqrt{2}E_8}$ is an irreducible $V_{\sqrt{2}E_8}$-module. If $\langle \alpha, \alpha\rangle = 1$, then we know that $\tau_{\hat{e}}(e^\alpha) = -e^{-\alpha}$ by Lemma 4.3. Thus $\tau_{\hat{e}}\sigma\tau_{\hat{e}}(e^\alpha) = \sigma^{-1}(e^\alpha)$ and so

$$\begin{aligned}\tau_{\hat{e}}\sigma\tau_{\hat{e}}(v_n e^\alpha) &= (\tau_{\hat{e}}\sigma\tau_{\hat{e}}(v))_n\,(\tau_{\hat{e}}\sigma\tau_{\hat{e}}(e^\alpha))\\ &= \sigma^{-1}(v)_n\sigma^{-1}(e^\alpha)\\ &= \sigma^{-1}(v_n e^\alpha)\end{aligned}$$

for any $v \in V_{\sqrt{2}E_8}$ by Lemma 4.1.

If $\langle \alpha, \alpha\rangle = 2$, then $\alpha = a + b$ for some vectors a, b in the forms of (4.1) with $\langle a, a\rangle = \langle b, b\rangle = 1$ and $\langle a, b\rangle = 0$. In this case, $e^\alpha = (e^a)_{-1}e^b$ and we still have $\tau_{\hat{e}}\sigma\tau_{\hat{e}}(e^\alpha) = \sigma^{-1}(e^\alpha)$. Thus for any $v \in V_{\sqrt{2}E_8}$,

$$\tau_{\hat{e}}\sigma\tau_{\hat{e}}(v_n e^\alpha) = \sigma^{-1}(v)_n\sigma^{-1}(e^\alpha) = \sigma^{-1}(v_n e^\alpha)$$

as required. □

As a consequence, we have the following proposition.

Proposition 4.5. *For any $u \in V_{\mathcal{L}}$, $\tau_{\hat{e}}\sigma\tau_{\hat{e}}(u) = \sigma^{-1}(u)$. Hence $\tau_{\hat{e}}\tau_{\hat{f}} = (\sigma^{-1})^2 = e^{2\pi\sqrt{-1}\beta(0)}$ as automorphisms of $V_{\mathcal{L}}$.*

Now we discuss the situation in the Leech lattice VOA V_Λ. First let us recall the following theorem [5, Theorem 4.1] (see also [10, 13]).

Theorem 4.6. *For any even unimodular lattice N of rank 24, there is at least one (in general many) isometric embedding of $\sqrt{2}N$ into the Leech lattice Λ.*

It is well known (cf. [10]) that the Leech lattice Λ can be constructed by "Construction A" for $\mathbb{Z}_4$-codes of length 24. In fact,

$$\Lambda = A_4(\mathcal{C}) = \frac{1}{2}\{x \in \mathbb{Z}^{24} \,|\, x \equiv c \mod 4 \quad \text{for some } c \in \mathcal{C}\}$$

for some type II self-dual $\mathbb{Z}_4$-code $\mathcal{C}$ of length 24. By [10], $\mathcal{C}$ can be taken to be the $\mathbb{Z}_4$-code having the generating matrix (4.2).

$$\begin{pmatrix}
2222 & 0000 & 0000 & 0000 & 0000 & 0000\\
0022 & 2200 & 0000 & 0000 & 0000 & 0000\\
0000 & 0022 & 2020 & 0000 & 0000 & 0000\\
0000 & 0000 & 0202 & 2020 & 0000 & 0000\\
0000 & 0000 & 0000 & 0202 & 2002 & 0000\\
2020 & 2020 & 0000 & 0000 & 0000 & 0000\\
0000 & 0220 & 2200 & 0000 & 0000 & 0000\\
0000 & 0000 & 2002 & 2002 & 0000 & 0000\\
0000 & 0000 & 0000 & 0022 & 2020 & 0000\\
2000 & 2000 & 2000 & 2000 & 2000 & 2000\\
1111 & 1111 & 2000 & 2000 & 0000 & 0000\\
2000 & 1111 & 1111 & 0000 & 2000 & 0000\\
0000 & 0000 & 1111 & 1111 & 2000 & 2000\\
2000 & 0000 & 2000 & 1111 & 1111 & 0000\\
2000 & 2000 & 0000 & 0000 & 1111 & 1111\\
3012 & 1010 & 1001 & 1001 & 1100 & 1100\\
3201 & 1001 & 1100 & 1100 & 1010 & 1010
\end{pmatrix} \tag{4.2}$$

For any $\mathbb{Z}_4$-code C of length n, one can obtain a binary code

$$B(C) = \{(b_1, \dots, b_n) \in {\mathbb{Z}_2}^n \mid (2b_1, \dots, 2b_n) \in C\},$$

where $2b_j$ should be considered as $0 \in \mathbb{Z}_4$ if $b_j = 0 \in \mathbb{Z}_2$ and $2 \in \mathbb{Z}_4$ if $b_j = 1 \in \mathbb{Z}_2$. Moreover, the lattice

$$L_{B(C)} = \{x \in \mathbb{Z}^n \mid x \in B(C) \mod 2\}$$

is a sublattice of $A_4(C)$. In the case for $C = \mathcal{C}$, the binary code $B(\mathcal{C})$ contains a subcode isomorphic to $H_8 \oplus H_8 \oplus H_8$. Thus by (3.11), we have an explicit embedding of $\sqrt{2}{E_8}^3$ into the Leech lattice Λ.

Now let $\sqrt{2}{E_8}^3 \longrightarrow \Lambda$ be any embedding of $\sqrt{2}{E_8}^3$ into the Leech lattice $\Lambda \subset \mathcal{L}^3$. Let $\tilde{\boldsymbol{\beta}} = \sqrt{2}(\mathbf{a}, 0, 0) \in \mathcal{L}^3$, where $\mathbf{a}$ is defined as in (3.6). Define $\tilde{\sigma} : (V_{\mathcal{L}})^{\otimes 3} \to (V_{\mathcal{L}})^{\otimes 3}$ by

$$\tilde{\sigma} = \sigma \otimes 1 \otimes 1 = e^{-\pi\sqrt{-1}\tilde{\boldsymbol{\beta}}(0)}.$$

Then $\tilde{\sigma}$ is an automorphism of V_Λ. Moreover, the following theorem holds.

Theorem 4.7. *Let $\tilde{\boldsymbol{\beta}}$ and $\tilde{\sigma}$ be defined as above. Then as automorphisms of V_Λ, $\tau_{\hat{e}}\tau_{\hat{f}} = (\tilde{\sigma}^{-1})^2 = e^{2\pi\sqrt{-1}\tilde{\boldsymbol{\beta}}(0)}$ and $|\tau_{\hat{e}}\tau_{\hat{f}}| = n_i$ for any $i = 0, 1, \dots, 8$.*

5. Correspondence with Conway's axes.

Recall the elements $\tilde{\omega}^k$ and X^j defined by (3.9) and (3.13). It turns out that the Griess algebra U_2 of U is generated by $\hat{e}$ and $\hat{f}$ and is of dimension $l + n_i - 1$ with basis $\tilde{\omega}^k, 1 \le k \le l$ and $X^j, 1 \le j \le n_i - 1$ (see [12] for details). We can verify that the Griess algebra U_2 coincides with the algebra described in Conway [1, Table 3]. In [1], it is shown that for each $2A$-involution of the Monster simple group, there is a unique idempotent in

the Monstrous Griess algebra $V_2^\natural$ corresponding to the involution. Such an idempotent is called an axis. By Miyamoto [15], an axis is exactly half of a conformal vector of central charge $1/2$. Note that the product $t * t'$ and the inner product (t, t') of two axes t, t' in [1] are equal to $t \cdot t' = t_1 t'$ and $\langle t, t' \rangle /2$, respectively in our notation. Let t_n be as in [1]. We denote t, u, v, and w of [1] by t_{2A}, u_{3A}, v_{4A}, and w_{5A}, respectively.

In each of the nine cases, we obtain an isomorphism of our Griess algebra U_2 to Conway's algebra generated by two axes through the following correspondence between our conformal vectors and Conway's axes.

$1A$ case. $\hat{e} \longleftrightarrow \frac{1}{32} t_0$.

$2A$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ j = 0, 1, \quad \tilde{\omega}^1 \longleftrightarrow \frac{1}{32} t_{2A}$.

$3A$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ j = 0, 1, 2, \quad \tilde{\omega}^1 \longleftrightarrow \frac{1}{45} u_{3A}$.

$4A$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ 0 \le j \le 3, \quad \tilde{\omega}^1 \longleftrightarrow \frac{1}{96} v_{4A}$.

$5A$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ 0 \le j \le 4, \quad \tilde{\omega}^1 - \tilde{\omega}^2 \longleftrightarrow -\frac{1}{35\sqrt{5}} w_{5A}$.

$6A$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ 0 \le j \le 5, \quad \tilde{\omega}^2 \longleftrightarrow \frac{1}{32} t_{2A}, \quad \tilde{\omega}^1 \longleftrightarrow \frac{1}{45} u_{3A}$.

$4B$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ 0 \le j \le 3, \quad \tilde{\omega}^1 \longleftrightarrow \frac{1}{32} t_{2A}$.

$2B$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ j = 0, 1$.

$3C$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ j = 0, 1, 2$.

References

1. J. H. Conway, A simple construction for the Fisher-Griess Monster group, *Invent. Math.* **79** (1985), 513–540.
2. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *ATLAS of Finite Groups*, Oxford Univ. Press, 1985.
3. J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag, Berlin-New York, 1988.
4. C. Dong, Vertex algebras associated with even lattices, *J. Algebra* **161** (1993), 245–265.
5. C. Dong, H. Li, G. Mason and S. P. Norton, Associative subalgebras of Griess algebra and related topics, *Proc. of the Conference on the Monster and Lie algebra at the Ohio State University*, May 1996, ed. by J. Ferrar and K. Harada, Walter de Gruyter, Berlin-New York, 1998, pp. 27–42.
6. I. B. Frenkel, Y. Huang and J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, Mem. Amer. Math. Soc. **104**, 1993.
7. I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Math., Vol. **134**, Academic Press, New York, 1988.
8. G. Glauberman and S. P. Norton, On McKay's connection between the affine E_8 diagram and the Monster, *CRM Proceedings and Lecture Notes*, Vol. **30**, Amer. Math. Soc., Providence, RI, 2001, pp. 37–42.
9. R. Griess, The Friendly Giant, *Invent. Math.* **69** (1982), 1–102.
10. M. Harada and M. Kitazume, $\mathbb{Z}_4$-code constructions for the Niemeier lattices and their embeddings in the Leech lattice, *Europ. J. Combinatorics* **21** (2000), 473–485.
11. C. H. Lam and H. Yamada, Decomposition of the lattice vertex operator algebra $V_{\sqrt{2}A_l}$, *J. Algebra* **272** (2004), 614–624.
12. C. H. Lam, H. Yamada and H. Yamauchi, McKay's observation and vertex operator algebras generated by two conformal vectors of central charge $1/2$, in preparation.

13. J. Lepowsky and A. Meurman, An E_8-approach to the Leech lattice and the Conway group, *J. Algebra* **77** (1982), 484–504.
14. J. McKay, Graphs, singularities, and finite groups, *Proc. Symp. Pure Math.*, Vol. **37**, Amer. Math. Soc., Providence, RI, 1980, pp. 183–186.
15. M. Miyamoto, Griess algebras and conformal vectors in vertex operator algebras, *J. Algebra* **179** (1996), 523–548.
16. M. Miyamoto, Binary codes and vertex operator (super)algebras, *J. Algebra* **181** (1996), 207–222.
17. M. Miyamoto, A new construction of the Moonshine vertex operator algebras over the real number field, to appear in *Ann. of Math.*
18. A. B. Zamolodchikov and V. A. Fateev, Nonlocal (parafermion) currents in two dimensional conformal quantum field theory and self-dual critical points in $\mathbb{Z}_N$-symmetric statistical systems, *Sov. Phys. JETP* **62** (1985), 215–225.

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